JOURNAL OF APPROXIMATION THEORY **90**, 75–87 (1997)
ARTICLE NO. AT963066

Exact Descriptions of Some K and E Functionals

Stefan Ericsson

Department of Mathematics, Luleå University, S-971 87 Luleå, Sweden
E-mail: sen@sm.luth.se

Communicated by Z. Ditzian

Received August 14, 1995; accepted in revised form June 19, 1996

Exact formulas for the K functional of some quasi-Banach couples of the type (X, L_∞) are obtained. In particular we can cover the cases when $X = L_{p,q}$ (*Lorentz* spaces), $0 < p < \infty$, $1 \leqslant q \leqslant \infty$, and $X = M_\varphi$ (*Marcinkiewicz* spaces). We also describe the K functional of the pair $(L_{p/q,1}, L_{p,q})$, $1 < q \leqslant p < \infty$. This generalizes the Nilsson–Peetre description of $K(t, f, L_1, L_q)$, see [11]. An optimal slicing technique gives an exact description of the E functional of the pair $(L_{p/q}, L_{p,q})$.

1. INTRODUCTION

The K functional plays a fundamental role in the theory of interpolation. There are only a few cases where exact formulas are known, e.g. for the couple (L_1, L_∞) we have the (Peetre) formula

$$K(t, f, L_1, L_\infty) = \int_0^t f^*(s)\, ds. \tag{1}$$

On the other hand, there are several well-known approximative formulas, see e.g. the bibliography by Maligranda [9]. However, it is of great interest to find exact descriptions. In this paper we present some formulas of this type.

Another reason for the importance of the K functional is its close connection to the E functional, which has applications to approximation theory. At the end of this paper we present an example how one can determine the E functional when the optimal decomposition for the corresponding K functional is known.

The K functional for the pair (X_0, X_1) of quasi-normed spaces, both linearly embedded in a vector space X, is defined for all $f \in X_0 + X_1$ and $t > 0$ as

$$K(t, f, X_0, X_1) := \inf\{\|g\|_{X_0} + t\,\|h\|_{X_1} \mid f = g + h,\ g \in X_0 \text{ and } h \in X_1\}.$$

0021-9045/97 $25.00

As infimum over affine mappings the function $K(\cdot, f, X_0, X_1)$ is concave. The E functional for the pair (X_0, X_1) of quasi normed spaces, both linearly embedded in a vector space X, is defined for all $f \in X_0 + X_1$ and $t > 0$ as

$$E(t, f, X_0, X_1) := \inf\{ \|f - g\|_{X_0} \mid f - g \in X_0,\ g \in X_1 \text{ and } \|g\|_{X_1} \leqslant t\},$$

where, as usual, $\inf \phi$ is defined as $+\infty$. It is well known that $E(\cdot, f, X_0, X_1)$ is a decreasing proper convex function when X_i are normed spaces. In the normed space case we redefine, if necessary, it at the point where it jumps to infinity so it will be lower semicontinuous there and hence everywhere.

For a function $f: (0, \infty) \to [0, \infty]$ we define, for all positive t,

$$f^{\bullet}(t) := \inf_{s>0} \{ f(s) + st \} \tag{2}$$

and

$$f^{\circ}(t) := \sup_{s>0} \{ f(s) - st \}. \tag{3}$$

These transformations are similar to the Legendre transform. We note that in the same manner as in the Legendre transform case, one sees that $f^{\bullet\circ} =: f^{\vee}$ is the greatest lower semicontinuous decreasing convex minorant of f and that $f^{\circ\bullet} =: f^{\wedge}$ is the least concave majorant of f. The following relations between the K and E functionals hold, (see [3, 13]):

$$K(\cdot, f, X_0, X_1) = E(\cdot, f, X_0, X_1)^{\bullet} \tag{4}$$

and

$$E(\cdot, f, X_0, X_1)^{\wedge} = K(\cdot, f, X_0, X_1)^{\circ}. \tag{5}$$

Note that $E(\cdot, f, X_0, X_1) = E(\cdot, f, X_0, X_1)^{\vee}$ in the case when X_i are normed spaces.

Unless something else is explicitly said we are working on a σ-finite measure space (Ω, Σ, μ). Throughout a primed exponent denotes the conjugate exponent.

The paper is organized as follows. We begin by determining the K functional for the pair $(L_{p,q}, L_\infty)$, $0 < p < \infty$ and $1 < q < \infty$ (see Theorem 2). This is done by actually finding the optimal level for the horizontal slicing. We know that the best decomposition is a horizontal slicing since the pair is a function lattice together with L_∞. We proceed by proving an exact formula of the K functional for the pair (X, L_∞), (see Theorem 3) where X is a quasi-normed function lattice satisfying an additional property. This property can be verified in many cases (see e.g. Corollaries 4 and 5). In particular we obtain an exact formula for (*weak*-L_p, L_∞).

In the last section we study the pair $(L_{p/q,1}, L_{p,q})$, $1<q\leqslant p<\infty$. As a special case, when $p=q$, we get the Nilsson–Peetre description of the K functional for the pair (L_1, L_q). The proof presented here is different from the one in [11]. In some cases, when one knows the optimal decomposition for the K functional, the E functional can be determined by using the Legendre type transform that describes the relationship between the E and K functionals. We finish this paper by demonstrate this technique for the couple $(L_{p/q,1}, L_{p,q})$, thereby obtaining the previously announced exact description of $E(t, f, L_{p/q,1}, L_{p,q})$.

2. EXACT FORMULAS FOR SOME K FUNCTIONALS OF THE TYPE $K(\cdot, \cdot, X, L_\infty)$

We start by pointing out a lemma of independent interest. This result appears in [4, 12] in the case $X=L_1$. The proof in the general case when X is a quasi-normed function lattice is similar but we present the details for the readers convenience. This lemma will be used to determine the K functional for some pairs of the type (X, L_∞).

LEMMA 1. *Let X be a quasi normed function lattice. Then, for $f\in X+L_\infty$,*

$$K(t, f, X, L_\infty)=\inf\{\|(|f|-\lambda)_+\|_X+t\lambda \mid \lambda\geqslant 0, (|f|-\lambda)_+\in X\}. \tag{6}$$

Proof. Consider an arbitrary decomposition $f=f_0+f_1$ with $f_0\in X$ and $f_1\in L_\infty$. Let

$$g:=|f|-\min\{\|f_1\|_\infty, |f|\}=(|f|-\|f_1\|_\infty)_+.$$

It follows that $g\leqslant|f_0|$ a.e. and, thus, $g\in X$ and

$$\|f_0\|_X+t\,\|f_1\|_\infty\geqslant\|g\|_X+t\,\|f_1\|_\infty.$$

Hence

$$K(t, f, X, L_\infty)\geqslant\inf\{\|(|f|-\lambda)_+\|_X+t\lambda \mid \lambda\geqslant 0, (|f|-\lambda)_+\in X\}.$$

In order to prove the inequality in the reversed direction we take $\lambda\geqslant 0$ with $(|f|-\lambda)_+\in X$ arbitrarily. Define $f_1:=h\min\{|f|, \lambda\}$, where $h(x)=0$ if $f(x)=0$ and $h(x)=f(x)/|f(x)|$ otherwise. Let $f_0:=f-f_1$. This gives $f_0=h(|f|-\lambda)_+$ and, hence, $f_0\in X$. Since $\|f_1\|_\infty\leqslant\lambda$ we have

$$K(t, f, X, L_\infty)\leqslant\inf\{\|(|f|-\lambda)_+\|_X+t\lambda \mid \lambda\geqslant 0, (|f|-\lambda)_+\in X\}.$$

This completes the proof. ∎

If we define $\|f\|_X$ as $+\infty$ when f is not an element of X, then (6) can be written as

$$K(t, f, X, L_\infty) = \inf_{\lambda > 0} \{ \|(|f| - \lambda)_+\|_X + \lambda t \}.$$

It is in this form it will be used in the sequel. We also note that Lemma 1 follows by using the well-known formula $E(t, f, X, L_\infty) = \|(|f| - \lambda)_+\|_X$, see [10], and the transform (4).

The *Lorentz* space $L_{p,q}$, $0 < p < \infty$, $0 < q < \infty$, is defined as the set of all real or complex valued measurable functions f such that

$$\|f\|_{p,q} := \left(\int_0^\infty [s^{1/p} f^*(s)]^q \frac{ds}{s} \right)^{1/q} < \infty,$$

where f^* denotes the rearrangement of f defined via

$$d_f(\lambda) := \mu\{x \in \Omega \mid |f(x)| > \lambda\}, \qquad \lambda \geqslant 0,$$

$$f^*(t) := \inf\{\lambda \geqslant 0 \mid d_f(\lambda) \leqslant t\}, \qquad t \geqslant 0.$$

The space $L_{p,q}$ will be a quasi-Banach function lattice with $\|\cdot\|_{p,q}$ as a quasi-norm.

For $0 < p < \infty$, $1 < q < \infty$ we let f denote an arbitrary function in $L_{p,q} + L_\infty$ and define

$$T(\lambda) := \frac{\|(|f| - \lambda)_+\|_{p/q', q-1}}{\|(|f| - \lambda)_+\|_{p,q}}, \tag{7}$$

for $\lambda \in (\underline{\lambda}, \|f\|_\infty)$ where

$$\underline{\lambda} := \inf\{\lambda \geqslant 0 \mid \|(|f| - \lambda)_+\|_{p,q} < \infty\}.$$

We note the following facts:

(a) $\underline{\lambda} < \infty$. This fact follows since $f \in L_{p,q} + L_\infty$ and the infimum is taken over a nonempty set.

(b) The function T is well defined. To see this we have to conclude that the numerator is finite. If $A_\alpha := \{x > 0 \mid f^*(x) > \alpha\}$ for $\alpha > \underline{\lambda}$, then $\nu(A_\alpha) < \infty$, where $d\nu = x^{(q/p) - 1}\, dx$. Indeed, take an arbitrary $\delta \in (\underline{\lambda}, \alpha)$, and we find that

$$\infty > \|(|f| - \delta)_+\|_{p,q}^q \geqslant \int_{A_\alpha} s^{q/p} (f^*(s) - \delta)^q \frac{ds}{s} \geqslant (\alpha - \delta)^q\, \nu(A_\alpha).$$

This implies that

$$\|(|f|-\alpha)_+\|_{p/q',\,q-1}^{q-1}=\int_0^\infty s^{q/p}(f^*(s)-\alpha)_+^{q-1}\frac{ds}{s}<\infty.$$

(c) The interval $(\underline{\lambda}, \|f\|_\infty)$ may be empty. T is obviously a continuous function, so the range of T is an interval.

The optimal level of the horizontal slicing in the following theorem is exactly what we get by putting the formal derivative equal to zero and solve for the level of slicing.

THEOREM 2. *Let* $0<p<\infty$, $1<q<\infty$ *and* $f\in L_{p,q}+L_\infty$. *If* $\underline{\lambda}<\|f\|_\infty$, *then*

$$K(t, f, L_{p,q}, L_\infty)$$
$$=\begin{cases}\|(|f|-\underline{\lambda})_+\|_{p,q}+t\underline{\lambda}, & t^{1/(q-1)}>T(\lambda)\quad \text{for all } \lambda\in(\underline{\lambda}, \|f\|_\infty)\\ \|(|f|-\lambda)_+\|_{p,q}+t\lambda, & t^{1/(q-1)}=T(\lambda)\\ t\,\|f\|_\infty, & t^{1/(q-1)}<T(\lambda)\quad \text{for all } \lambda\in(\underline{\lambda}, \|f\|_\infty).\end{cases}$$

In the remaining case, when $\underline{\lambda}=\|f\|_\infty$, *we have* $K(t, f, L_{p,q}, L_\infty)=t\,\|f\|_\infty$.

Proof. Let $F(\xi):=\|(|f|-\xi)_+\|_{p,q}+t\xi$. According to Lemma 1 we know that the K functional for the pair $(L_{p,q}, L_\infty)$ is equal to $\inf\{F(\xi)\,|\,\xi\geqslant 0\}$. Furthermore, it is clear that it suffices to take the infimum over $F(\xi)$ where $\xi\in[\underline{\lambda}, \|f\|_\infty]$.

Assume that $\underline{\lambda}<\|f\|_\infty$. Choose $\xi\in[\underline{\lambda}, \|f\|_\infty]$ arbitrarily and let $t^{1/(q-1)}=T(\lambda)$, $d:=\max\{\lambda, \xi\}$ and $A_\alpha:=\{x>0\,|\,f^*(x)>\alpha\}$. We have

$$\begin{aligned}&[F(\xi)-F(\lambda)]\,\|(|f|-\lambda)_+\|_{p,q}^{q-1}\\ &\quad=(\xi-\lambda)\,\|(|f|-\lambda)_+\|_{p/q',\,q-1}^{q-1}-\|(|f|-\lambda)_+\|_{p,q}^{q}\\ &\qquad+\|(|f|-\xi)_+\|_{p,q}\,\|(|f|-\lambda)_+\|_{p,q}^{q-1}.\end{aligned}$$

By applying the Hölder inequality on the last term in the right hand side we find that

$$\begin{aligned}&[F(\xi)-F(\lambda)]\,\|(|f|-\lambda)_+\|_{p,q}^{q-1}\\ &\quad\geqslant(\xi-\lambda)\,\|(|f|-\lambda)_+\|_{p/q',\,q-1}^{q-1}-\|(|f|-\lambda)_+\|_{p,q}^{q}\\ &\qquad+\int_{A_\delta}s^{q/p}(f^*(s)-\xi)(f^*(s)-\lambda)^{q-1}\frac{ds}{s}.\end{aligned}$$

Therefore, since the integrals over A_δ on the right hand side vanishes,

$$[F(\xi)-F(\lambda)]\ \|(|f|-\lambda)_+\|_{p,q}^{q-1}$$

$$\geqslant(\xi-\lambda)\int_{A_\lambda\setminus A_\delta} s^{q/p}(f^*(s)-\lambda)^{q-1}\,\frac{ds}{s}-\int_{A_\lambda\setminus A_\delta} s^{q/p}(f^*(s)-\lambda)^{q}\,\frac{ds}{s}$$

$$\geqslant(\xi-\lambda)\int_{A_\lambda\setminus A_\delta} s^{q/p}(f^*(s)-\lambda)^{q-1}\,\frac{ds}{s}$$

$$-(\delta-\lambda)\int_{A_\lambda\setminus A_\delta} s^{q/p}(f^*(s)-\lambda)^{q-1}\,\frac{ds}{s}=0.$$

We conclude that $F(\xi)$ attains minimum at $\xi=\lambda$.

If $t^{1/(q-1)}>T(\lambda)$ for all $\lambda\in(\underline{\lambda}, \|f\|_\infty)$ it follows that F is strictly increasing on $(\underline{\lambda}, \|f\|_\infty)$. Indeed, let $\delta<\xi$ and use the Hölder inequality to find that

$$[F(\xi)-F(\delta)]\ \|(|f|-\delta)_+\|_{p,q}^{q-1}$$

$$=\|(|f|-\delta)_+\|_{p,q}^{q-1}\,((\xi-\delta)\,t+\|(|f|-\xi)_+\|_{p,q}-\|(|f|-\delta)_+\|_{p,q})$$

$$>\|(|f|-\delta)_+\|_{p,q}^{q-1}\,((\xi-\delta)\,T(\delta)^{q-1}+\|(|f|-\xi)_+\|_{p,q}-\|(|f|-\delta)_+\|_{p,q})$$

$$\geqslant\int_{A_\delta} s^{q/p}(f^*(s)-\delta)^{q-1}\,[\xi-\delta+f^*(s)-\xi-f^*(s)+\delta]\,\frac{ds}{s}=0.$$

Hence

$$K(t, f, L_{p,q}, L_\infty)=\lim_{\xi\downarrow\underline{\lambda}} F(\xi)=\|(|f|-\underline{\lambda})_+\|_{p,q}+t\underline{\lambda}.$$

In a similar way we find that if $t^{1/(q-1)}<T(\lambda)$ for all $\lambda\in(\underline{\lambda}, \|f\|_\infty)$, then F is strictly decreasing on $(\underline{\lambda}, \|f\|_\infty)$. If $\xi\uparrow\|f\|_\infty$, then $F(\xi)\downarrow t\,\|f\|_\infty$. Hence

$$K(t, f, L_{p,q}, L_\infty)=t\,\|f\|_\infty.$$

The remaining case, i.e. when $\underline{\lambda}=\|f\|_\infty$, is trivial. ∎

Remark 1. Theorem 2 can immediately be generalised to the case where $L_{p,q}$ is substituted by the more general space $L_{\varphi,q}$. The space $L_{\varphi,q}$ is defined as the set of real or complex valued μ measurable functions f such that

$$\|f\|_{\varphi,q}:=\left(\int_0^\infty [f^*(s)\,\varphi(s)]^q\,\frac{ds}{s}\right)^{1/q}<\infty,$$

where φ is quasi concave. In this case the function T will be

$$T(\lambda) := \frac{\|(|f|-\lambda)_+\|_{\varphi^{q'}, q-1}}{\|(|f|-\lambda)_+\|_{\varphi, q}}.$$

Next we state the following generalization of the Peetre formula (1):

THEOREM 3. *Let X be a quasi normed function lattice and let f be a fixed function in $X+L_\infty$. Assume that X has the following property: There exists a function B_f such that*

$$\|(|f|-\lambda)_+\|_X = \sup_{\delta>0} \{B_f(\delta)-\delta\lambda\}, \tag{8}$$

holds for all $\lambda>0$. Then

$$K(\cdot, f, X, L_\infty) = B_f^\wedge, \tag{9}$$

where $B_f^\wedge$ denotes the least concave majorant of B_f. Conversely, for (9) to holds at t it is necessary that X satisfies (8) for $\lambda = t$.

Remark 2. $\|f\|_X$ is assumed to be $+\infty$ when $f \notin X$. For the last part of the theorem the function $x \mapsto \|(|f|-x)_+\|_X$ is assumed to be lower semicontinuous. Since it is convex the only possibility for non lower semicontinuity is at the point where it jumps to infinity, if necessary we redefine it there. This will not be necessary if X has the Fatou property, i.e. if $0 \leqslant f_n \uparrow f$ μ-a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$.

Proof. According to Lemma 1 and the properties of the transforms defined by (4) and (5) it yields that

$$\begin{aligned} K(t, f, X, L_\infty) &= \inf_{\lambda>0} \{\|(|f|-\lambda)_+\|_X + \lambda t\} \\ &= \inf_{\lambda>0} \{\sup_{\delta>0} \{B_f(\delta)-\delta\lambda\} + \lambda t\} = B_f^{\circ\bullet}(t) = B_f^\wedge(t). \end{aligned}$$

Conversely, take the transform $g \mapsto g^\circ$ on

$$\inf_{\lambda>0} \{\|(|f|-\lambda)_+\|_X + \lambda t\} = \inf_{\lambda>0} \{\sup_{\delta>0} \{B_f(\delta)-\delta\lambda\} + \lambda t\}$$

and observe that $x \mapsto \|(|f|-x)_+\|_X$ is convex and lower semicontinuous. ∎

When we say that a space X has the property (8) we mean that (8) is satisfied for all f in $X+L_\infty$. By using Theorem 3 in concrete cases we obtain as special cases both well-known and new exact formulas for the K

functional. In particular, we believe that the formulas in the following two corollaries are new.

COROLLARY 4. *Let* $0 < p < \infty$ *and* $f \in L_{p,\infty} + L_\infty$. *Then*

$$K(t, f, L_{p,\infty}, L_\infty) = (tf^*(t^p))^\wedge. \tag{10}$$

Proof. We observe that $L_{p,\infty}$ has the property (8) with $B_f(t) = tf^*(t^p)$. Indeed, it yields that

$$\begin{aligned} \|(|f|-\lambda)_+\|_{p,\infty} &= \sup_{\delta>0} \{\delta^{1/p}(|f(\delta)|-\lambda)^*_+\} \\ &= \sup_{\delta>0} \{\delta^{1/p}(f^*(\delta)-\lambda)\} \\ &= \sup_{\delta>0} \{\delta f^*(\delta^p) - \delta\lambda\}. \end{aligned}$$

This completes the proof. ∎

Remark 3. For the case $p = 1$, (10) appears without proof in [5]. For $p \neq 1$ the following equivalence formula

$$K(t, f, L_{p,\infty}, L_\infty) \sim \sup_{s \leqslant t^p} s^{1/p} f^*(s)$$

is known. This is a special case of the Holmstedt formula, see [5]. Moreover, one easily sees that, for all $t > 0$,

$$\sup_{s \leqslant t^p} s^{1/p} f^*(s) \leqslant (tf^*(t^p))^\wedge \leqslant 2 \sup_{s \leqslant t^p} s^{1/p} f^*(s).$$

Next we consider the *Marcinkiewicz* space M_φ. Let φ be a quasi-concave function on $\mathbb{R}^+ := (0, \infty)$, i.e. a positive increasing function such that $\varphi(t)/t$ is decreasing. The space M_φ consists of all real or complex valued μ-measurable functions f defined on Ω for which

$$\|f\|_{M_\varphi} := \sup_{t>0} \left\{ \frac{\varphi(t)}{t} \int_0^t f^*(s)\, ds \right\}$$

is finite. With $\|\cdot\|_{M_\varphi}$ as norm M_φ is a Banach function lattice.

COROLLARY 5. *Let* φ *be a quasi concave bijection on* $\mathbb{R}^+$. *For* $t > 0$ *and* $f \in M_\varphi + L_\infty$ *we have*

$$K(t, f, M_\varphi, L_\infty) = \left(\frac{t}{\varphi^{-1}(t)} \int_0^{\varphi^{-1}(t)} f^*(s)\, ds \right)^\wedge.$$

Proof. Since $\varphi(t)$ tends to zero as $t \downarrow 0$ it yields that

$$\|(|f|-\lambda)_+\|_{M_\varphi} = \sup_{\delta>0}\left\{\frac{\varphi(\delta)}{\delta}\int_0^\delta (f(s)-\lambda)_+^*\,ds\right\}$$
$$= \sup_{\delta>0}\left\{\frac{\varphi(\delta)}{\delta}\int_0^\delta f^*(s)\,ds - \varphi(\delta)\,\lambda\right\}$$
$$= \sup_{\delta>0}\left\{\frac{\delta}{\varphi^{-1}(\delta)}\int_0^{\varphi^{-1}(\delta)} f^*(s)\,ds - \delta\lambda\right\}.$$

Therefore the property (8) holds with

$$B_f(t) = \frac{t}{\varphi^{-1}(t)}\int_0^{\varphi^{-1}(t)} f^*(s)\,ds,$$

and the proof is complete. ∎

As a special case we get the (Peetre) formula (1). We just choose φ as the identity function and observe that $t \mapsto \int_0^t f^*(s)\,ds$ is concave.

There are many more examples of spaces that satisfy (8), e.g. $L_{p,1}$. We only need to observe that $\|f\|_{p,1} = \sup_\delta \int_0^\delta s^{1/p} f^*(s)(ds/s)$ and the formula

$$K(t, f, L_{p,1}, L_\infty) = \int_0^{(t/p)^p} s^{1/p} f^*(s)\,\frac{ds}{s}$$

follows if $p \geqslant 1$ and if $p < 1$ the formula is correct if we take the operator on the right hand side. Another way of arriving at this formula, when $p \geqslant 1$, is by using the well-known formula, see [7, 14]:

$$K(t, f, \Lambda_\varphi, \Lambda_\psi) = \int_0^\infty f^*\,d\min(\varphi, t\psi).$$

Note that $L_{p,1} \equiv \Lambda_\varphi$ with $\varphi(t) = pt^{1/p}$ and $L_\infty \equiv \Lambda_\psi$ with $\psi = \chi_{(0,\infty)}$, where χ denotes the characteristic function.

Yet another example of a space that satisfies (8) is $L_1 + L_\infty$. This is a *Marcinkiewicz* space. We have $L_1 + L_\infty \equiv M_{\min(1,\cdot)}$, but since $\min(1,\cdot)$ is not a bijection Corollary 5 is not applicable. To verify that $M_{\min(1,\cdot)}$ has the property (8) we discuss as in the proof of Corollary 5 and use the following equality

$$\sup_{\delta>0}\left\{\frac{\min(1,\delta)}{\delta}\int_0^\delta f^*(s)\,ds - \min(1,\delta)\,\lambda\right\} = \sup_{\delta>0}\left\{\int_0^{\min(1,\delta)} f^*(s)\,ds - \delta\lambda\right\}.$$

One obtains the well-known formula, see [8],

$$K(t, f, L_1+L_\infty, L_\infty)=\int_0^{\min(1,t)} f^*(s)\,ds.$$

We end this section by giving just one example of the fact that this technique also can give equivalence formulas for the K functional. Maligranda showed in [8], that if X is a rearrangement invariant space on the semiaxis $(0, \infty)$ and $\varphi(t) := \|\chi_{(0,t)}\|_X$, then $K(\varphi(t), f, X, L_\infty(0, \infty))$ is equivalent to $\|f^*\chi_{(0,t)}\|_X$. This can also be seen (in the spirit of Theorem 3) in the following way:

$$\begin{aligned}
K(\varphi(t), f, X, L_\infty(0,\infty)) &= \inf_{\lambda>0} \{\|(f^*-\lambda)_+\|_X+\lambda\varphi(t)\}\\
&\geqslant \inf_{\lambda>0} \{\sup_{\delta>0} \{\|(f^*-\lambda)_+\chi_{(0,\delta)}\|_X+\lambda\varphi(t)\}\}\\
&\geqslant \inf_{\lambda>0} \sup_{\delta>0} \{\|f^*\chi_{(0,\delta)}\|_X-\lambda\varphi(\delta)+\lambda\varphi(t)\}\\
&\geqslant \inf_{\lambda>0} \{\|f^*\chi_{(0,t)}\|_X-\lambda\varphi(t)+\lambda\varphi(t)\}\\
&= \|f^*\chi_{(0,t)}\|_X.
\end{aligned}$$

To prove $K(\varphi(t), f, X, L_\infty(0,\infty)) \leqslant 2\,\|f^*\chi_{(0,t)}\|_X$ we use the fact that the sum of rearrangement invariant spaces is rearrangement invariant, see e.g. [6]. Hence

$$\begin{aligned}
K(\varphi(t), f, X, L_\infty(0,\infty)) &= K(\varphi(t), f^*, X, L_\infty(0,\infty))\\
&\leqslant \|f^*\chi_{(0,t)}\|_X+\varphi(t)\,\|f^*\chi_{[t,\infty)}\|_\infty\\
&= \|f^*\chi_{(0,t)}\|_X+\|\chi_{(0,t)}\|_X\,f^*(t)\\
&\leqslant 2\,\|f^*\chi_{(0,t)}\|_X,
\end{aligned}$$

and we have proved that $K(\varphi(t), f, X, L_\infty(0,\infty)) \approx \|f^*\chi_{(0,t)}\|_X$ with equivalence constants 1 and 2.

3. DESCRIPTIONS OF THE K AND E FUNCTIONALS FOR THE PAIR $(L_{p/q,1}, L_{p,q})$

We describe the K functional for the pair $(L_{p/q,1}, L_{p,q})$, $1<q\leqslant p<\infty$. This is a generalization of the description of $K(t, f, L_1, L_q)$ by Nilsson and Peetre in [11], see also [1]. The proof here is somewhat different from the ones in [11] and [1]. By using this result, we can also obtain an exact description of the E functional $E(t, f, L_{p/q,1}, L_{p,q})$.

For $f \in L_{p/q,1} + L_{p,q}$, $1 < q \leqslant p < \infty$ we define, for $\lambda > 0$,

$$G(\lambda) := \frac{p}{q} (d_f(\lambda))^{q/p} + \int_{d_f(\lambda)}^{\infty} (s^{1/p} f^*(s))^q / \lambda^q \frac{ds}{s}.$$

THEOREM 6. *For $t > 0$ and $f \in L_{p/q,1} + L_{p,q}$, $1 < q \leqslant p < \infty$,*

$$K(t, f, L_{p/q,1}, L_{p,q}) = \begin{cases} \int_0^{d_f(\lambda)} s^{q/p} f^*(s) \frac{ds}{s} + \int_{d_f(\lambda)}^{\infty} (s^{1/p} f^*(s))^q / \lambda^{q-1} \frac{ds}{s}, & t^{q'} = G(\lambda) \\ \|f\|_{p/q,1}, & t^{q'} \geqslant \frac{p}{q} (d_f(0))^{q/p} \end{cases}.$$

Moreover, the optimal decomposition is given by a horizontal slicing.

Remark 4. The properties of G implies that the above description covers all cases.

Proof. The function G is decreasing and continuous. Furthermore $G(\lambda) \to 0$ as λ tends to infinity and $G(\lambda) \to p/q(d_f(0))^{q/p}$ as λ tends to zero. Hence the equation $t^{q'} = G(\lambda)$ can be solved when $t^{q'} < (p/q)(d_f(0))^{q/p}$. Let λ satisfy $t^{q'} = G(\lambda)$. Then, we have

$$\begin{aligned} K(t, f, L_{p/q,1}, L_{p,q}) &\leqslant \|(|f| - \lambda)_+\|_{p/q,1} + t \, \|\min(|f|, \lambda)\|_{p,q} \\ &= \int_0^{d_f(\lambda)} s^{q/p} f^*(s) \frac{ds}{s} + \int_{d_f(\lambda)}^{\infty} (s^{1/p} f^*(s))^q / \lambda^{q-1} \frac{ds}{s} \\ &= \int_0^{\infty} f^*(s) \, g(s) \, ds, \end{aligned}$$

where

$$g(s) := \begin{cases} s^{(q/p)-1}, & s < d_f(\lambda) \\ s^{(q/p)-1} f^*(s)^{q-1} / \lambda^{q-1}, & s \geqslant d_f(\lambda). \end{cases}$$

Choose an arbitrary decomposition $f = f_0 + f_1$ with $f_0 \in L_{p/q,1}$ and $f_1 \in L_{p,q}$. Use the Hardy lemma, see e.g. [2] p. 56, and we see that

$$\int_0^{\infty} f^*(s) \, g(s) \, ds \leqslant \int_0^{\infty} f_0^*(s) \, g(s) \, ds + \int_0^{\infty} f_1^*(s) \, g(s) \, ds.$$

Apply the Hölder inequality to find that

$$\int_0^\infty f^*(s)\, g(s)\, ds \leqslant \|f_0\|_{p/q,\,1} + t\,\|f_1\|_{p,\,q}.$$

Take infimum over all decompositions and the formula

$$K(t, f, L_{p/q,\,1}, L_{p,\,q}) = \int_0^{d_f(\lambda)} s^{q/p} f^*(s)\,\frac{ds}{s} + \int_{d_f(\lambda)}^\infty (s^{1/p} f^*(s))^q/\lambda^{q-1}\,\frac{ds}{s}$$

follows.

Let now $t^{q'} \geqslant (p/q)(d_f(0))^{q/p}$. In particular this implies that $f \in L_{p/q,\,1}$ and, hence, $K(t, f, L_{p/q,\,1}, L_{p,\,q}) \leqslant \|f\|_{p/q,\,1}$. When $t^{q'}$ tends to $(p/q)(d_f(0))^{q/p}$ we see that λ tends to zero. But

$$\lim_{\lambda \to 0} \left(\int_0^{d_f(\lambda)} s^{q/p}\, f^*(s)\,\frac{ds}{s} + \int_{d_f(\lambda)}^\infty (s^{1/p} f^*(s))^q/\lambda^{q-1}\,\frac{ds}{s} \right) = \|f\|_{p/q,\,1},$$

and we conclude that $K(t, f, L_{p/q,\,1}, L_{p,\,q}) = \|f\|_{p/q,\,1}$. We note that in both cases the optimal decomposition is a horizontal slicing and the proof is complete. ▮

In order to determine $E(t, f, L_{p/q,\,1}, L_{p,\,q})$ one could try to apply formula (5) but this is not at all easy. Instead one should try to write the K functional as a transform like (4) and then using the transform (5).

THEOREM 7. *Let $t > 0$ and $f \in L_{p/q,\,1} + L_{p,\,q}$, $1 < q \leqslant p < \infty$. We have*

$$E(t, f, L_{p/q,\,1}, L_{p,\,q}) = (\|(|f| - \delta(t))_+\|_{p/q,\,1})^\vee,$$

where

$$\delta(t) := \begin{cases} g^{-1}(t), & t < \|f\|_{p,\,q} \\ \|f\|_\infty, & t \geqslant \|f\|_{p,\,q}, \end{cases}$$

with $g(t) := \|\min(|f|, t)\|_{p,\,q}$.

Proof. From the previous theorem we know that

$$K(t, f, L_{p/q,\,1}, L_{p,\,q}) = \inf_{\lambda \geqslant 0} \{\|(|f| - \lambda)_+\|_{p/q,\,1} + t\,\|\min(|f|, \lambda)\|_{p,\,q}\}.$$

Obviously g is a continuous bijection from $(0, \|f\|_\infty)$ to $(0, \|f\|_{p,\,q})$. Hence

$$K(t, f, L_{p/q,\,1}, L_{p,\,q}) = \inf_{\lambda > 0} \{\|(|f| - \delta(\lambda))_+\|_{p/q,\,1} + t\lambda\}$$
$$= (\|(|f| - \delta(t))_+\|_{p/q,\,1})^\bullet.$$

In view of (5) it follows

$$E(t, f, L_{p/q,1}, L_{p,q}) = (\|(|f| - \delta(t))_+\|_{p/q,1})^{\vee}.$$

This completes the proof. ∎

Remark 5. The technique presented in the proof of Theorem 7 can be used in several different cases when one knows the decomposition that obtains the infimum in the definition of the K functional, e.g. for the pairs $(L_1^{\omega_0}, L_1^{\omega_1})$ and $(\Lambda_\varphi, \Lambda_\psi)$.

ACKNOWLEDGMENT

The author thanks the Professors Lech Maligranda and Lars-Erik Persson for their generous advice and encouragement, which have improved the final version of this paper.

REFERENCES

1. J. Bastero, Y. Raynaud, and M. L. Rezola, An explicit expression for the K_r functionals of interpolation between L^p spaces, *Publ. Mat.* **35** (1991), 97–117.
2. C. Bennett and R. Sharpley, "Interpolation of Operators," Academic Press, Boston, 1988.
3. Yu. A. Brudnyi and N. Ya. Krugljak, "Interpolation Functors and Interpolation Spaces," North-Holland, Amsterdam, 1991.
4. P. L. Butzer and H. Behrens, "Semi-Groups of Operators and Approximation," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
5. T. Holmstedt, Interpolation of quasi-normed spaces, *Math. Scand.* **26** (1970), 177–199.
6. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, "Interpolation of Linear Operators" Transl. Math. Monographs, Vol. 54, Amer. Math. Soc., Providence, RI, 1982.
7. G. G. Lorentz and T. Shimogaki, Interpolation theorems for operators in function spaces, *J. Funct. Anal.* **2** (1968), 31–51.
8. L. Maligranda, "The K-Functional for Symmetric Spaces," Lecture Notes in Math., Vol. 1070, pp. 169–182, Springer-Verlag, New York/Berlin, 1984.
9. L. Maligranda, Interpolation of operators and application, 1926–1990 (bibliography), Dept. of Math., Luleå University, 1990.
10. L. Maligranda and L. E. Persson, The E-functional for some pairs of groups, *Res. Mat.* **20** (1991), 538–553.
11. P. Nilsson and J. Peetre, On the K functional between L^1 and L^2 and some other K functionals, *J. Approx. Theory* **48** (1986), 322–327.
12. E T. Oklander, Interpolación, espacios de Lorentz y teorema de Marcinkiewicz, Fasc. 20, Univ. Buenos Aires, 1995.
13. J. Peetre and G. Sparr, Interpolation of normed abelian groups, *Ann. Pura Appl.* **92** (1972), 217–262.
14. R. Sharpley, Space $\Lambda_\alpha(X)$ and interpolation, *J. Funct. Anal.* **11** (1972), 479–513.